De Broglie geometry eliminating the infinities of QED; An exact derivation of the Lamb shift formula in the normal case

Z. I. Szabó[***]

Abstract

This paper evolves a new non-perturbative theory by which the problem of infinities appearing in quantum physics can be handled. Its most important application is an exact derivation of the Lamb shift formula by using no renormalization. The Lamb shift experiment (1947) gave rise to one of the greatest challenges whose explanation brought the modern renormalization technique into life. Since then this is the only tool for handling these infinities. The relation between this renormalization theory and our non-perturbative theory is also discussed in this paper.

Our key insight is the realization that the natural complex Heisenberg group representation splits the Hilbert space, $L^2_{\mathbb{C}}(\mathbb{R}^{2\kappa})$, of complex valued functions defined on an even dimensional Euclidean space into irreducible subspaces (alias zones) which are invariant also under the action of the Landau-Zeeman operator. After a natural modification, also the Coulomb operator can be involved into this zonal theory. Thus these zones can be separately investigated, both from geometrical and physical point of view. In the literature only the zone spanned by the holomorphic polynomials has been investigated so far. This zone is the well known Fock space. This paper explicitly explores also the ignored (infinitely many) other zones. It turns out that quantities appearing as infinities on the total Hilbert space are finite in the zonal setting. Even the zonal Feynman integrals are well defined. In a sense, the desired finite quantities are provided here by an extended particle theory where these extended objects show up also on the rigorously developed mathematical level. Name *de Broglie geometry* was chosen to suggest this feature of the zonal theory.

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[**]Lehman College of CUNY, Bronx, NY 10468,USA, and Rényi Institute, Budapest, POBox 127, 1364 Hungary. E-mail: zoltan.szabo@lehman.cuny.edu

1 Introduction.

This paper consists of three parts. In the first two chapters all those mathematical and physical structures are reviewed which are most essential to the third one entitled "Interaction with the Coulomb field".

In the *first chapter* the Zeeman-Hamilton operator, determined for a system of free charged particles orbiting in constant magnetic fields, is established as the Laplacian on a Riemannian, so called, Zeeman manifold. As far as the author knows, this exact matching of a physical Hamiltonian with the Laplacian of a Riemann manifold has never been recognized in the literature so far. Although the manifolds with such coincidence can be introduced in a most general way, in this paper only those defined by center periodic metric 2-step nilpotent Lie groups, $\Gamma_Z\backslash N$, are considered. In this formula the $\Gamma_Z = \{Z_\gamma\}$ is a partial lattice defined only on the center $\mathbf{z}$ of the metric group (N, g), where g is a left-invariant Riemann metric on N. The factor manifold is then a trivial torus bundle, $\mathbf{R}^k \times T^l$, on which the metric can be briefly described as follows. Both on the base $\mathbf{R}^k$ and the torus $T^l = \Gamma_Z\backslash\mathbf{z}$, the induced metrics are flat (Euclidean), but, at a point (X, Z), the tangent spaces $T_X(\mathbf{R}^k)$ and $T_Z(T^l)$ are not perpendicular. This property shows that the product $\times$ is just topological and not metric. In this paper mostly Heisenberg-type groups (see the definition later) will be considered. This particular groups have even dimensional X-spaces, $\mathbf{R}^k = \mathbf{R}^{2\kappa}$, for which the dimension, l, of the Z-space can be arbitrary natural number.

The natural L^2 Hilbert space defined on the torus bundle is subjugated, first, to a *primary splitting* $L^2_{\mathbb{C}}(\mathbf{R}^k \times T^l) = \sum_\gamma W_\gamma$. Then, in the *secondary splitting*, each subspace W_γ is further decomposed into the zones indicated in the Abstract. The primary splitting is nothing but the Fourier-Weierstrass decomposition by means of the Γ_Z-periodic Fourier functions $e^{2\pi\langle Z_\gamma, Z\rangle\mathbf{i}}$ defined by means of the lattice points $Z_\gamma \in \Gamma_Z$ on the torus T^l. The above L^2 Hilbert space on the total space decomposes into orthogonal subspaces, W_γ, spanned by functions of the form $\Psi(X, Z) = \psi(X)e^{2\pi\langle Z_\gamma, Z\rangle\mathbf{i}}$, where $\psi(X)$ is an L^2 function defined on $\mathbf{R}^k$.

The Riemannian Laplacian, Δ, on the total space is described in formula (2). This is not yet the Zeeman operator, however, its action on a function Ψ can be written in the form $\Delta\Psi_\gamma = \Box_\gamma(\psi)(X)e^{2\pi\langle Z_\gamma, Z\rangle\mathbf{i}}$, i. e., it leaves each W_γ invariant and induces action only on function $\psi(X)$ depending just on X. This latter operator is explicitly described in formula (3). Comparison with (1) reveals that operator $-(1/2)\Box_\gamma$ is nothing but a Zeeman operator satisfying $V = 0$. Although it is acting on functions depending just on the space-variables, the $\Box_\gamma$ is not a sub-Laplacian. In fact, it is obtained on the

invariant subspaces defined on the total space and not by the submersion of the total space onto the base $\mathbf{R}^k$. The 2D version of this Hamiltonian was introduced by Landau, in 1928. By our construction, it is obtained on a 3 dimensional time-periodic Heisenberg group endowed with the natural left-invariant metric. In this paper, the explicit computations will be demonstrated mostly on this 2D version. The general theory is developed in [Sz6].

There is pointed out in Section2.3 that this model is in strong relationship with Dirac's those relativistic multi-time model, where the relativity is furnished by attributing self-times to the particles. (This theory differs from his relativistic electron theory.) Our model is attached to $\kappa = k/2$ particles orbiting in complex planes in constant magnetic fields. The latter fields pin down unique inertia systems which define the self-times of the particles. The time coordinates measured in this process appear on the center of the group, however, the time is a secondary concept in this theory. The primary objects are the angular momentum endomorphisms defining the constant magnetic fields.

The second theme of the paper involves and further develops the Fock-Bargmann representation of the complex Heisenberg groups. From now on the investigations are performed on the X-space, $\mathbb{R}^k$. They are understandable without the mathematical model described above and it can be considered as spectral analysis of the Zeeman operator. It is connected with the above model such that the invariant subspaces W_γ defined by the primary decomposition are further investigated, which, by the map $\Psi_\gamma(X, Z) \to \psi(X)$, can be identified with the function space $L^2_{\mathbf{C}}(X)$ defined just on the X-space. By the above discussions, the Laplacian Δ appears on a fixed invariant subspace as the Landau-Zeeman operator $\Box_\lambda$ acting on this function space. The zones are defined by a spectral decomposition of $L^2_{\mathbf{C}}(X)$, thus it can be regarded as a secondary decomposition of the subspaces obtained in the primary decomposition.

This *secondary splitting* is defined by the Fock-Bargmann representation of the complex Heisenberg group. In the literature, this representation is considered only on the Fock space, generated by the holomorphic polynomials in the total space of complex valued functions defined on $\mathbf{R}^k$. No other invariant subspaces of this reducible representation have been investigated so far. This paper explores all irreducible subspaces, called Zeeman zones, of the FB-representation. This zonal spectral analysis includes the explicit description of various zonal objects such as the projection kernels, the zonal spectra, and the zonal Wiener- resp. Schrödinger-flows. The most surprising result is that both zonal flows are of the trace class, defining the zonal

partition and zeta functions in the standard way without renormalization. Even the zonal Feynman measures on the path-space are well defined. In other words, quantities appearing as infinities on the global level are well defined finite ones in the zonal setting.

The third and most important chapter includes the (so far ignored) Coulomb operator V in the investigations. Since it does not commute with the Zeeman-Landau operator, this operator defines transmissions $V^{(a,b)}$: $\mathcal{H}^{(a)} \to \mathcal{H}^{(b)}$ between the zones. Such a map is defined by projecting the product $V\varphi^{(a)}$, where $\varphi^{(a)} \in \mathcal{H}^{(a)}$, onto $\mathcal{H}^{(b)}$. Then $V^{(a,a)}$ maps $\mathcal{H}^{(a)}$ onto itself and is called zonal Coulomb operator. In order to keep the zones invariant, only these zonal Coulomb operators are retained. The complete zonal Zeeman operators are defined by $-(1/2)\Box^{(a)} - V^{(a,a)}$ and the other transmission operators are omitted. The most remarkable features of the zonal Coulomb operators are that they commute with the zonal Zeeman-Landau operator and their discrete spectrum appears on eigenfunctions which are common with the Landau-Zeeman operator. Furthermore, they become integral operators exhibiting local interactions. This phenomenon is to the contrary of the global interaction characteristic for the total operator V. However, both the trace and L^2-norm of $V^{(a,a)}$ are infinities. This means that this operator still defines infinite energy summations and is not yet suitable for analyzing the Lamb shift.

In a simple trace-computation the eigenvalues are summed up without any probabilistic distinction (equipartition principle) which often has caused problems and false conclusions in quantum physics. In many cases these problems are solved by finding an appropriate finite measure by which the energy summations should be implemented. For instance, in the Planck hypothesis, which concerns the amount of energy $U(\nu)d\nu$ radiated by the blackbody in the frequency range between ν and $\nu + d\nu$, beside the quantized energy there is assumed also the existence of a natural probability which determines the likelihood that the blackbody emits-absorbs a certain energy. By supposing equal likelihood for radiating $U(\nu)$ (equipartition principle), the old theory yielded the Rayleigh-Jeans law: $U(\nu)d\nu \sim \kappa T\nu^2 d\nu$, which contradicts the empirical curve described by the Wien law: $U(\nu)d\nu \sim \nu^3 e^{-h\nu/\kappa T}d\nu$. The primary evidence justifying the Planck hypothesis was that it yielded the Wien law.

An analysis of Bethe's classical paper, written for explaining Lamb shift, shows that also his starting formula determining the energy due to the interaction of the electron with the radiation field assumes equipartition principle. The applications of cutoff constants and other renormalization techniques can be interpreted as a process which turns this summation into a proba-

bilistic one. It is apparent that the solution of this problem requires finding an adequate finite complex measure which describes the considered interaction on a probabilistic background. This amplitude is explicitly constructed by means of the spectrum of $V^{(a,a)}$, which, in its final form, appears in terms of the Gamma function $\Gamma(z)$. The natural computation yielding the Lamb shift formula in the last section is the major evidence which justifies that this amplitude most adequately describes the interaction of the electron with the Coulomb field on the quantum level.

Since this non-perturbative computation leads to the very same formula, the renormalization technique seems to have expended most of its effort to take off the transmission operators $V^{(a,b)}$ defined for $a \neq b$ from the Coulomb field and establish, by means of the remaining $V^{(a,a)}$, an energy summation which has probabilistic features. The question arises if these omitted transmission operators are existing real physical objects? Before answering this question note that these operators are of zero trace class satisfying $V^{(a,b)} = \overline{V}^{(b,a)}$. These properties mean that the same amount of energy is transported from $\mathcal{H}^{(a)}$ to $\mathcal{H}^{(b)}$ as from $\mathcal{H}^{(b)}$ to $\mathcal{H}^{(a)}$. Thus there is no real energy transmission provided by these operators. What are then these transmission operators for? We suggest the following answer to this problem: The transmission operators are real existing parts of the Coulomb operator whose main role is to maintain the zonal structure. Without their action all the zones and zonal electrons would blow up. This explanation gives a satisfactory answer for the most difficult question arising in all theories which work with extended particles: Why do the spread-out zonal charged particles not blow up?

2 Zeeman manifolds

2.1 Zeeman-Hamilton operators.

The classical Zeeman operator of a charged particle is

$$H_Z = -\frac{\hbar^2}{2m_r}\Delta + \frac{\hbar eB}{2m_r c\mathbf{i}}D_z \bullet + \frac{e^2B^2}{8m_r c^2}(x^2+y^2) + V, \tag{1}$$

where $V = -Z_p e^2/r$ is the Coulomb potential originated from the nucleus formed by Z_p protons, furthermore, $m_r = m_e m_p/(m_e+m_p)$ is the reduced mass which differs from the mass, m_e, of the electron just by a small amount (in case of the hydrogen atom, $(m_e - m_r)/m_e \approx 5 \times 10^{-4}$). Because of the small difference, we use $m_e = 9.1093818810^{-28}g$ also in the place of m_r.

This operator is written up in Gaussian (centimeter-gram-second) units, where the Bohr magneton is defined by $\mu_B = e\hbar/2m_e c$ and the unit charge, esu of charge or statcoulomb, is defined so that the Coulomb force constant is 1. In SI (m-kg-s) units the Bohr magneton is $\mu_B = e\hbar/2m_e = 9.27400949(80)10^{-24} JT^{-1}$, where the magnetic field has SI units of tesla, $1T = 1kgs^{-1}C^{-1}$ and symbol C, Coulomb, denotes unit of electric charge. The corresponding operator in SI units omits the light speed $c = 299792458$ m/s from the second and third term of (1) and the Coulomb force constant is $k_C \approx 8.98810^9 N \cdot m^2 \cdot C^{-2}$. Later computations mostly prefer the SI units. The classical papers quoted later use the Gaussian units, however, and this is why the units will be used in a mixed way.

This operator is usually considered on the 3-space. The free particle operator restricted onto the (x, y)-plane (i. e., $V = 0$ and Δ is the Euclidean Laplacian on $\mathbf{R}^2$) is called Landau Hamiltonian. This paper proceeds with this 2D-version and its generalizations defined on complex vector spaces $\mathbf{C}^{k/2} = \mathbf{R}^k$. Operator $D_z\bullet = x\partial_y - y\partial_x$, called angular momentum operator, commutes with the remaining part, $\mathbf{O} = H_Z - \frac{\hbar eB}{2m_r c\mathbf{i}} D_z\bullet$, of the complete Hamiltonian, thus the spectrum appears on common eigenfunctions. I. e., the $D_z\bullet$ splits the spectral lines of $\mathbf{O}$, which phenomena is associated with the Zeeman effect. Actually, the H_Z is the Hamilton operator of an electron orbiting about the origin of the (x, y)-plane in a constant magnetic field $\mathbf{K} = B\partial_z$. Since it is still revolving around the origin, the free Landau particle is only latent-free. The 3D-version can be established by means of the Maxwell equations and the real Heisenberg group representation. To establish the Landau Hamiltonian, one can use the Fock-Bargmann representation of the complex Heisenberg group.

2.2 Mathematical modeling: Zeeman manifolds.

Interestingly enough, the Landau operator, H_Z, can be identified with the Laplace operators of two step nilpotent Lie groups endowed with the natural left invariant metrics. As far as the author knows, this interpretation is unknown in the literature. A 2-step nilpotent metric Lie group is defined on the product $\mathbf{v} \oplus \mathbf{z}$ of Euclidean spaces, where the components, $\mathbf{v} = \mathbf{R}^k$ and $\mathbf{z} = \mathbf{R}^l$, are called X- and Z-space respectively. The Lie algebra is completely determined by the linear space, $J_{\mathbf{z}}$, of skew endomorphisms acting on the X-space defined by $\langle [X, Y], Z\rangle = \langle J_Z(X), Y\rangle$, where $X, Y \in \mathbf{v}$ and J_Z is the endomorphism associated with $Z \in \mathbf{z}$. To be more precise, the Lie algebra is uniquely determined by the system $\{\mathbf{v} \oplus \mathbf{z}, \mathcal{A} : \mathbf{z} \to SE(\mathbf{v})\}$, where $\mathcal{A} : Z \to J_Z$ is a one-to-one linear map from $\mathbf{z}$ into the space of skew endomorphisms

acting on $\mathbf{v}$. If the range, $J_{\mathbf{z}}$, is the same for two maps, then they define isomorphic Lie algebras which are isometricly isomorphic if the combined map $\mathcal{A}_2^{-1}\mathcal{A}_1 : \mathbf{z} \to \mathbf{z}$ is orthogonal. The natural innerproduct on $\mathbf{z}$ is defined by $\langle Z_1, Z_2\rangle = -TrJ_{Z_1}J_{Z_2}$. The metric, g, is the left invariant extension of the natural Euclidean metric on the Lie algebra. The exponential map identifies the Lie algebra with the group. Thus also the group is defined on the same space of (X, Z)-vectors on which the Lie algebra is living.

Particular 2-step nilpotent Lie groups are the Heisenberg-type groups, introduced by Kaplan [K], defined by endomorphism spaces satisfying the Clifford condition $J_Z^2 = -|Z|^2 id$. These metric groups are attached to Clifford modules, thus the classification of these modules provides classification also for the H-type groups. In this case the X-space decomposes into the product $\mathbf{v} = (\mathbf{R}^{r(l)})^{a+b} = \mathbf{R}^{r(l)a} \times \mathbf{R}^{r(l)b}$ and endomorphisms J_Z are defined by endomorphisms j_Z acting on the smaller space $\mathbf{R}^{r(l)}$. Namely, the J_Z acts on $\mathbf{R}^{r(l)a}$ resp. $\mathbf{R}^{r(l)b}$ as $j_Z \times \cdots \times j_Z$ resp. $-j_Z \times \cdots \times -j_Z$. The H-type groups are denoted by $H_l^{(a,b)}$, indicating the above decomposition.

The Laplacians on H-type groups are of the form

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \sum_{\alpha=1}^{r} \partial_\alpha D_\alpha \bullet, \tag{2}$$

where $D_\alpha \bullet$ denotes directional derivatives along the fields $J_\alpha(X) = J_{Z_\alpha}(X)$ and $\{Z_\alpha\}$ is an orthonormal basis on the Z-space. This operator is not the Landau operator yet. It appears, however, on center periodic H-type groups, $\Gamma\backslash H$, defined by factorizing the center of the group with a Z-lattice $\Gamma = \{Z_\gamma\}$. In fact, in this case the L^2 function space is the direct sum of function spaces W_γ spanned by functions of the form $\Psi_\gamma(X, Z) = \psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$. Later on, the Fourier-Weierstrass decomposition $L^2_{\mathbb{C}}(\mathbb{R}^k \times T^l) = \sum_\gamma W_\gamma$ is called *primary splitting*. Each W_γ is invariant under the action of Δ. More precisely, $\Delta\Psi_\gamma(X, Z) = \Box_\gamma\psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$ holds where operator $\Box_\gamma$, acting on $L^2(\mathbf{v})$, is of the form

$$\Box_\gamma = \Delta_X + 2\pi\mathbf{i}D_\gamma \bullet -4\pi^2|Z_\gamma|^2(1 + \frac{1}{4}|X|^2). \tag{3}$$

When the invariant subspaces are defined by the functions $\Psi_\gamma(X, Z) = \psi(X)e^{-2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$, this operator appears in the following form:

$$\Box_\gamma = \Delta_X - 2\pi\mathbf{i}D_\gamma \bullet -4\pi^2|Z_\gamma|^2(1 + \frac{1}{4}|X|^2). \tag{4}$$

The first basic observation in this paper is that the Landau Hamiltonian satisfying $B = 1T$ and $|e| = 1.60217653 \times 10^{-19}C$ (elementary charge) can

be identified with $H_Z = -(\hbar^2/2m_r)\Box_\gamma$, where cases (3) and (4) correspond to $e > 0$ and $e < 0$ respectively. Furthermore, $\pi|Z_\gamma| = \lambda = |e|/2\hbar c$ holds, thus the λ^2 can be interpreted as the eigenvalue of $-\pi^2 J_\gamma^2$. Also note that switching the angular momentum endomorphism J_γ to $-J_\gamma$ transforms (4) to (3). This means that choosing the sign of the charge is equivalent to choosing J or $-J$ from the set of complex structures available on $\mathbf{R}^2$. Then the operators, regarding both for electron and positron, appear in the common form (3), where e is a positive quantity. Later on, this operator is denoted also by $\Box_\lambda$.

Comparing with the Landau operator, this one contains a surplus constant $4\pi^2|Z_\gamma|^2 = 4\lambda^2$, which contribution does not show up in the original Landau Hamiltonian. In a physical situation, one considers the $\hbar^2/2m_r$-times of this term. Then, by $m_r \approx m_e$, $\mu_B = 2\lambda\hbar^2/2m_e$, one has: $W_{extra} = (2m_e/\hbar^2)\mu_B^2 \approx 1.408970181 \times 10^{-8} kg \cdot s^{-2}T^{-2}$. Dimension $kg \cdot s^{-2} = J/m^2$ shows that this extra term is a constant energy density field whose integral on the whole space would provide infinite energy. Unit T^{-2} indicates that this energy is nothing but the self energy $q\int \mathbf{B}^2$ of the constant magnetic field. The exact numerical values for the other terms are: $2\lambda = 2m_e\mu_B/\hbar^2 \approx 1.492298399 \times 10^{15} m^{-2}T^{-1}$ and $\hbar^2/2m_e = 6.1042635 \times 10^{-39} kg \cdot m^2$. These values are used in the later computations.

2.3 Interpretations for the basic objects

For a $(k+1)$-dimensional Heisenberg group, defined by a complex structure J acting on the even dimensional Euclidean space $\mathbf{v} = \mathbf{R}^k$, number $k/2 = \kappa$ is interpreted as the number of particles. In view of the 2D Landau operator, this is the most natural interpretation for this number. Also the interpretation for the X-space $\mathbf{v} = \mathbf{R}^k$ is clear. It must be the space where the particles are orbiting in their own constant magnetic fields such that each particle occupies a complex plane. These interpretations can be carried over to general center periodic 2-step nilpotent Lie groups, where, due to the higher dimensional Z-space, the Zeeman operator appears in a more complicated form.

On Heisenberg groups,the center is interpreted as the non-relativistic time-axis but this concept can not be obviously taken over to the general cases. To the complete interpretation of the Z-space one must answer, first, that how is the time measured in these models? Equally important question is if this model is relativistic or non-relativistic?

The key to answering these questions is the realization that the Lie algebra is uniquely determined by the map $\mathcal{A} : Z \to J_Z$ which corresponds skew

endomorphisms to the elements of the center. Instead of time coordinates, this map a priori corresponds angular momentums which define constant magnetic fields and vanishing electric fields for the particles. But this particular appearance of a constant electromagnetic field pins down, on each complex plane occupied by a particle, a unique inertia system with well defined self-time, $t(J_Z)$, measured in the system. This denotation indicates that this self-time depends on the angular momentum endomorphism. Also note that the inertia system defined for proportional Z's are uniquely determined and the line spanned by Z and parameterized by the arc-length t can be interpreted as the self-time-axis in the inertia system.

This construction attributing self-times to the particles relates our model to Dirac's famous multi-time model, where, in order to establish a relativistic quantum theory, self-time is attributed to the particles. By this reason, our model can be called relativistic as far as Dirac's multi-time theory fits this characterization. It should be point out that this is not the classical relativism and for distinguishing from the original theory, this is called *anchored relativism*. This name was chosen to recall the key idea in the self-time construction: the constant electromagnetic field ($\mathbf{B}_\alpha = \mathbf{c}onstant \neq 0, \mathbf{E}_\alpha = 0$) defined by an angular momentum endomorphism J_Z "anchors" the system in a unique inertia system which defines the self-time for a particle.

This self-time construction can be implemented on the center of the non-periodic group, after which, operator (2) can be interpreted as a positive definite version of the Klein-Gordon operator corresponding to the anchored relativistic theory. This argument shows that this anchored relativism does not contradicts the original relativistic theory, however, it can not be identified with it either. For instance, unless introducing negative energies into the system, one can not transform operator (2) into the original (indefinite) Klein-Gordon operator just by changing the sign before Δ_Z to the minus one. An other distinguishing feature is that this model leads to probabilistic quantum theory working with positive probabilities defined just on the space. Recall that Dirac's relativistic electron theory (which is different from his above mentioned multi-time theory) establishes such positive probabilities on the Minkowski space-time. This idea was strongly criticized by Pauli, according to whom such probabilistic theory makes sense only on the space.

The periodic model can be regarded as a partial crystal model where the crystal is in the center of the group. The system can be in crystal states represented by the endomorphisms J_γ. Parameters $\lambda_i > 0$ are defined by the absolute values of the eigenvalues of πJ_γ appearing on the corresponding complex eigenplanes. Thus $|\pi Z_\gamma|^2 = \lambda_1^2 + \cdots + \lambda_\kappa^2$. The Hamilton operators

belonging to these crystal states are $-\frac{1}{2}\Box_\gamma$. In case of a single eigenvalue λ with multiplicity κ, the corresponding operator is denoted also by $\Box_\lambda$. Then $|\pi Z_\gamma|^2 = \kappa\lambda^2$ holds. This paper deals only with such systems.

2.4 Isospectrality constructions

The Riemannian manifolds considered here were originally used for constructing isospectral manifolds with different local geometries [Sz1]-[Sz4]. The isospectrality examples arouse on certain compact submanifolds both of the center-periodic and the non-periodic groups. Here only the examples constructed on Heisenberg-type groups $H_l^{(a,b)}$ will be explained. For a fixed l, what is the dimension of the center, consider all those groups for which also $(a + b)$ is the same value. All these metric groups live on the same manifold $\mathbf{R}^{r(l)a} \times \mathbf{R}^{r(l)b} \times \mathbf{R}^l$. The only difference between two groups in a family is exhibited by the endomorphisms J_Z which are defined by the endomorphisms j_Z acting on the smaller space $\mathbf{R}^{r(l)}$ such that the J_Z acts on $\mathbf{R}^{r(l)a}$ resp. $\mathbf{R}^{r(l)b}$ as $j_Z \times \cdots \times j_Z$ resp. $-j_Z \times \cdots \times -j_Z$. By the physical interpretation, the j_Z resp. $-j_Z$ correspond to positrons resp. electrons, therefore, these groups are attached to the same number of particles and the only distinguishing feature is the ratio of numbers of electrons and positrons. One can go from one system to the other by exchanging some of the electrons for positrons.

It is well known in physics that the spectrum does not change during electron-positron exchanges. A strict convert of this physical statement to a mathematical one is that the spectra computed on the whole non-compact center-periodic groups $\Gamma\backslash H_l^{(a,b)}$ for the members of the considered family are same. In the above papers this isospectrality is established in a much stronger form, namely, not just on the non-compact groups (which cover the physical cases) but also on a wide range of compact submanifolds such as ball×torus-, sphere×torus-, sphere×ball-, sphere×sphere-, ball-, and sphere-type submanifolds. Although the physical statement gives some chance for these much stronger isospectralities, these statements are rather non-trivial because a non-trivial isospectrality of two manifolds does not imply the isospectralities of the submanifolds.

There is an other very surprising statement established in the above papers. Namely, the members in an isospectrality family can have different local geometries. This statement is true, for instance, exactly for those families of Heisenberg-type groups where number l is of the form $l = 4r + 3$, where $r = 0, 1, \ldots$. The other families consist of isometricly isomorphic groups for which the isospectrality is a trivial statement.

We demonstrate the different local geometries for the members of the isospectrality family $H_3^{(a,b)}$, where the Z-space, $\mathbf{R}^3$, is identified with the space of imaginary quaternionic numbers, the "small" X-space is $\mathbf{R}^{r(l)} = \mathbf{R}^4 = \mathbf{H}$ (the space of quaternions), and the endomorphisms are defined by $j_Z(h) = Zh$. When the group is attached to the same type of particles (i.e., it is $H_3^{(a+b,0)}$ or $H_3^{(0,a+b)}$), then both the sphere×torus- and sphere×sphere-type submanifolds are homogeneous (the isometries act transitively), while for mixed particles, characterized by the relation $ab \neq 0$, these submanifolds are locally inhomogeneous. The existence of isospectral metrics having different local geometries is unknown even in physics where no geometries attached to spectra have been considered so far. This phenomenon certainly must have some effect on a deeper understanding of the symmetries on the quantum level, but this impact is not well understood yet.

3 Normal de Broglie Geometry

3.1 Introducing the zones.

In what follows, all investigations are performed on the X-space. Actually, the following parts are understandable without knowing about the mathematical model described above and it can be considered as spectral analysis of the Zeeman operator. It is connected with the above model such that the invariant subspaces W_γ defined by the primary decomposition are further investigated, which, by the map $\Psi_\gamma(X, Z) \to \psi(X)$, can be identified with the space $L^2_{\mathbf{C}}(X)$ consisting functions defined just on the X-space. The Laplacian Δ appears on a fixed invariant subspace as the Landau-Zeeman operator $\Box_\lambda$ which acts on complex valued functions defined on the X-space. The zones are defined by a spectral decomposition of $L^2_{\mathbf{C}}(X)$, thus it can be regarded as a secondary decomposition of the subspaces obtained in the primary decomposition. In this paper the endomorphism $-\pi^2 J_\gamma^2$ has only one eigenvalue λ^2 with multiplicity k, thus $|\pi J_\gamma|^2 = \kappa\lambda^2$ holds.

The Hilbert space, $\mathcal{H}$, of the complex valued L^2-functions is isomorphic to the weighted space defined by the Gauss density $d\eta_\lambda(X) = e^{-\lambda|X|^2} dX$. The latter space is spanned by the complex valued polynomials. Next $\mathcal{H}$ is considered in this form. The natural *complex Heisenberg group representation* on $\mathcal{H}$ is defined by

$$\rho_{\mathbf{c}}(z_i)(\psi) = (-\partial_{\overline{z}_i} + \lambda z_i \cdot)\psi \quad , \quad \rho_{\mathbf{c}}(\overline{z}_i)(\psi) = \partial_{z_i}\psi, \tag{5}$$

where $\{z_i\}$ is a complex coordinate system on the X-space. This representation is reducible. In fact, it is irreducible on the Fock space generated by the

holomorphic polynomials, where it is called Fock-Bargmann representation. Besides the Fock space there are infinitely many other irreducible invariant subspaces. By this reason, the above representation is called *extended Fock-Bargmann representation* and the irreducible decomposition defined by this representation is called *secondary splitting*. In the function operator correspondence, this representation associates operator (1) to the Hamilton function of an electron orbiting in a constant magnetic field.

The zones are defined in two different ways. First, they can be defined by the invariant subspaces of representation (5). The actual construction uses Gram-Schmidt orthogonalization. On the complex plane $\mathbf{v} = \mathbf{C}$, corresponding to the 2D Landau operator, the $\mathcal{H}$ is the direct sum of subspaces $G^{(a)}$ spanned by functions of the form $\overline{z}^a h$, where h is an arbitrary holomorphic polynomial. Then one gets the zones $\mathcal{H}^{(a)}$, where $a = 0, 1, 2, \dots$, by the Gram-Schmidt orthogonalization process applied to the function spaces $G^{(a)}$. It is clear that the first zone, $\mathcal{H}^{(0)}$, is the Fock space. The zone index a indicates the maximal number of the antiholomorphic coordinates $\overline{z}$ in the polynomials spanning the zone.

One of the referees of [Sz6] pointed out to me that the polynomials produced by this constructions were considered also by Itô [I] in the context of complex Markov processes. In fact, Itô defines the Hermite polynomials of complex variables for $p, q = 0, 1, 2, \dots$ by the explicit formula

$$H_{pq}(z, \overline{z}) = \sum_{s=0}^{min(p,q)} (-1)^s \frac{p!q!}{s!(p-s)!(q-s)!} z^{p-s} \overline{z}^{q-s}. \tag{6}$$

In the 2D case, they form an orthogonal basis in $\mathcal{H}$ defined for $\lambda = 1$. In this formalism, the zones are spanned by polynomials belonging to fixed values of q, i. e., the q corresponds to the zone index a in our formalism. Corresponding to the cases $p \geq q$ resp. $q \geq p$, these formulas appear in the form $f_n(r^2) z^{p-q}$ resp. $f_n(r^2) \overline{z}^{q-p}$, where $f_n(t)$ is a polynomial of order $n = min(p, q)$. These formulas will be reconstructed in Section 4 in terms of the Laguerre polynomials

$$L_n^{(l)}(t) = \sum_{i=0}^{n} \binom{n+l}{n-i} \frac{(-t)^i}{i!}, \tag{7}$$

where they appear in the form

$$(-1)^n n! L_n^{(l)}(r^2) z^l \quad \text{resp.} \quad (-1)^n n! L_n^{(l)}(r^2) \overline{z}^l. \tag{8}$$

Then, substitution $r^2 = z\overline{z}$ converts (8) to (6).

The construction with the Gram Schmidt orthogonalization easily extends to general dimensions. Gross zone $\mathcal{H}^{(a)}$ is constructed by means of all polynomials $\overline{z}_1^{(a_1)} \dots \overline{z}_{k/2}^{(a_{k/2})}$ satisfying $a_1 + \dots + a_{k/2} = a$. This gross zone is the direct sum of the subzones $\mathcal{H}^{(a_1 \dots a_{k/2})}$ defined for the particular values $a_1, \dots, a_{k/2}$. The eigenfunctions appear as an appropriate product of eigenfunctions defined for the complex coordinate planes (these details are mostly omitted in this review, but see further remarks in the end of this section). In the 2D-case all the zones are irreducible under the action of the extended Fock-Bargmann representation. In the higher dimensions, however, the *gross zones* are reducible and the subzones are irreducible. Note that the holomorphic (Fock) zone is always irreducible. For the sake of simplicity, all the formulas below are established on the gross zones.

In terms of Ito's polynomials (6), which are defined for arbitrary dimension k, the eigenfunctions appear in the form $h^{(p,\upsilon)}(X) = H^{(p,\upsilon)}(X)e^{-\lambda|X|^2/2}$ with the corresponding eigenvalues $-((4p+k)\lambda + 2k\lambda^2)$, where p resp. υ are the holomorphic resp. antiholomorphic degrees of polynomial $H^{(p,\upsilon)}$ (the last term is due to to the energy-density defined by the constant magnetic field where $4(k/2) = 2k$ applies). Numbers $\tau = p + \upsilon$ and $m = 2p - \tau$ are called *total-* and *magnetic quantum numbers* (TQN and MQN) respectively. The above function is an eigenfunction also of the magnetic dipole moment operator with eigenvalue m. Then a zone is spanned by eigenfunctions having the same index υ. According to the formula $\upsilon = \frac{1}{2}(\tau - m)$, the zones are determined by the quantum numbers τ and m. For a given τ, the range of m is $-l, -l+1, \dots, l-1, l$, where $l = |p - \upsilon|$, and the eigenfunctions belonging to different MQN's are sorted out to distinct zones. In this sense, a zone exhibits the magnetic state of the particle. Note that eigenvalues are independent of the antiholomorphic index and they depend just on the holomorphic index. As a result, each eigenvalue has infinite multiplicity. On the irreducible zones, however, each multiplicity is $k/2 = \kappa$. Moreover, two irreducible zones are isospectral.

It is important to understand that the above spectrum computation is not the standard one, in which case the eigenfunctions are sought in the form $f_{n,l}(|X|^2)G^{(l)}(X)e^{-\frac{1}{2}\lambda|X|^2}$, where $G^{(l)}$ is an l^{th}-order homogeneous harmonic polynomial and $f_{n,l}(t)$ is an n^{th}-order polynomial which depends also on l. These standard explicit eigenfunction computations are completely established in Section4.1. It was Schrödinger who computed the eigenfunctions and eigenvalues of his operators in this form. The classical quantum numbers are derived from this representation of the eigenfunctions. Note that formulas (8) represent the eigenfunctions exactly in this form. Indeed, in

the 2D case, the homogeneous harmonic polynomials are of the form z^l or $\overline{z}^l$ and functions $f_{n,l}$ happen to be the corresponding Laguerre polynomials. This conversion is such easy just in the 2D-case. In [Sz6], these computations are established for arbitrary dimensions, where 2 different type of eigenfunction computations are developed. One of them is of traditional (Schrödinger) type and the other seeks the eigenfunctions as the product of Itô's polynomials. According to the traditional terminology of spectroscopy, the azimuthal quantum number is defined by the order, l, of G and order $n(l)$ is the radial quantum number. Thus $\tau = l + 2n(l)$ and $p = n + l, \upsilon = n$ resp. $p = n, \upsilon = n + l$ hold, corresponding to the cases indicated in (8). The magnetic quantum numbers, m, are defined in both cases by the same numbers.

These formulas clearly describe the conversion of quantum numbers defined by the different representation of the eigenfunctions. However, the standard representation with radial functions and spherical harmonics eclipses the zonal structure even in the 2D case, which is, on the other hand, very clearly exhibited by the new type of technique also in general dimensions. In the zonal theory the Itô polynomial technique and not the standard one is the natural tool for developing a clear spectral analysis. This preference refers also to the quantum numbers defined by the two techniques. Probably these standard computational techniques constitute the main reasons for the zones have not been investigated in the literature earlier. Nor have the intriguing fact, asserting that functions (6) are eigenfunctions of the Landau Hamiltonian, been exploited so far.

3.2 Projection kernels and point-spreads.

In the literature only the projection onto the Fock space $\mathcal{H}^{(0)}$ is well known, which turned out to be a convolution operator with the so called Fock-Bargmann kernel $\left(\frac{\lambda}{\pi}\right)^{k/2} e^{\lambda(z\cdot\overline{w} - \frac{1}{2}(|z|^2+|w|^2)}$. Our theory, developed in [Sz5, Sz6], explicitly determines the projection also onto a general zone $\mathcal{H}^{(a)}$. Then the corresponding projection kernel is

$$\delta^{(a)}_{\lambda z}(w) = \left(\frac{\lambda}{\pi}\right)^{\frac{k}{2}} L_a^{(\frac{k}{2}-1)}(\lambda|z-w|^2) e^{\lambda(z\cdot\overline{w} - \frac{1}{2}(|z|^2+|w|^2)}, \tag{9}$$

where $L_a^{((k/2)-1)}(t)$ is the Laguerre polynomial indicated by the indexes. To have this formula, consider an orthonormal basis $\{\varphi_i^{(a)}\}_{i=1}^{\infty}$ formed by eigenfunctions being in $\mathcal{H}^{(a)}$. The projection kernel can be formally expressed in the form $2\delta^{(a)}(z,w) = \sum_i \varphi_i^{(a)}(z)\overline{\varphi_i^{(a)}(w)}$, where z and w represent complex

vectors on $\mathbf{C}^\kappa = \mathbf{R}^k$. Then the formula can be established by means of the explicit eigenfunctions. These kernels can be interpreted as restrictions of the global Dirac delta distribution, $2\delta_z(w) = \sum \varphi_i(z)\overline{\varphi}_i(w)$, onto the zones.

These kernels represent one of the most important concepts in this theory. They can be interpreted such that, on a zone, a point particle appears as a spread described by the above wave-kernel. Note that how these kernels, called zonal point-spreads, are derived from the one defined for the holomorphic (Fock) zone. This holomorphic spread, which involves a Gauss function, is just multiplied by the radial Laguerre polynomial corresponding to the zone. This form of the functions describing the point-spreads show the most definite similarity to the de Broglie wave packets. In a rigorous theory, function $\delta^{(a)}_{\lambda Z}\overline{\delta}^{(a)}_{\lambda Z}$ is the density of the point-spread concentrated around Z and $\delta^{(a)}_{\lambda Z}$ is the so called spread-amplitude. On a given zone the point-spreads are the most compressed wave packets, yet they are distributed all over the whole space. This zonal particle theory gives a clear explanation for the Aharanov-Bohm (AB) effect [AB] as well as other phenomenas described in [Sz5].

The AB effect produces relative phase shift between two electron beams enclosing a magnetic flux even if they do not touch the magnetic field. This effect has no explanation in the classical mechanics and it contradicts even the relativistic principle of *all fields must interact only locally.* Yet, this effect was clearly demonstrated by the *Tonomura et al experiments* [T1, T2].

Although the point electrons do not touch the fields, the vector potential involved into the Hamilton operator of the system does reach there. Exploiting this phenomena, Aharanov and Bohm explained the effect by the "significance of electromagnetic potentials in the quantum theory". In classical physics this potential is considered to be a mere mathematical convenience which is completely meaningless from physical point of view. In de Broglie geometry the zonal particles are extended ones which must touch the magnetic field, which is a clear enough explanation for the AB effect. Since the zones are defined by a particular vector potential, this explanation is in accordance with the Aharanov-Bohm idea. Indeed, the vector potential is not just a mathematical convenience any more but it is one of the important physical objects by which the zonal structure is defined.

Despite that the experiments were performed under the condition of complete confinement of the magnetic field in the magnet, some physicists have questioned the validity of the tests, attributing the phase shift to leakage fields. The electron spread idea developed in this paper can be interpreted such that not the magnetic field but "the electrons are leaking".

3.3 Global Wiener- and Schrödinger-flows.

The zonal analysis is continued in this section by describing the global flows defined on the total Hilbert space $\mathcal{H}$. Because of its trivial contribution to the formulas, the surplus constant belonging to the constant magnetic field is omitted and H_Z means the Landau Hamiltonian. The global Wiener-flow, $e^{-tH_Z}(t,X,Y)$, appears in the following explicit form:

$$\big(\frac{\lambda}{2\pi sinh(\lambda t)}\big)^{k/2} e^{-\lambda(\frac{1}{2}coth(\lambda t)|X-Y|^2+\mathbf{i}\langle X,J(Y)\rangle}. \tag{10}$$

This kernel satisfies the Chapman-Kolmogorov identity and it tends to $\delta(X,Y)$ when $t\to 0_+$. However, it is not of the trace class, thus functions such as the partition function or the zeta function are not defined in the standard way. Also note that by regularization (renormalization) only well defined relative(!) partition and zeta functions are introduced.

The global Schrödinger kernel, $e^{-t\mathbf{i}H_Z}(t,X,Y)$, appears in the following explicit form:

$$\big(\frac{\lambda}{2\pi\mathbf{i}sin(\lambda t)}\big)^{k/2} e^{\mathbf{i}\lambda\{\frac{1}{2}cot(\lambda t)|X-Y|^2-\langle X,J(Y)\rangle\}}. \tag{11}$$

Since for fixed t and X the function depending on Y is not L^2, the integral required for the Chapman-Kolmogorov identity is not defined for this kernel. Neither is this kernel of the trace class. Nevertheless, it satisfies the above limit property when $t\to 0_+$.

It is well known that rigorously defined measure on the path-spaces can be introduced only with the Wiener kernel e^{-tH}. Note that the heat kernel involves a Gauss density which makes this constructions possible. Whereas, the Schrödinger kernel does not involve such term. This is why no well defined constructions can be carried out with this kernel. These difficulties disappear, however, by considering these constructions on the zones separately.

3.4 Zonal Wiener- and Schrödinger-flows.

The zones are invariant with respect to the action of the Hamilton (Laplace) operator, thus the zonal flows are well defined on each zone. The zonal Wiener-kernels are of the trace class, which can be described by the following explicit formulas.

$$e^{-tH_Z^{(0)}} = \big(\frac{\lambda e^{-\lambda t}}{\pi}\big)^{\frac{k}{2}} e^{\lambda(-\frac{1}{2}(|X|^2+|Y|^2)+e^{-2\lambda t}\langle X,Y+\mathbf{i}J(Y)\rangle)}, \tag{12}$$

$$e^{-tH_Z^{(a)}} = \mathcal{L}_a^{(\frac{k}{2}-1)}(t,X,Y))e^{-tH_Z^{(0)}}(t,X,Y), \tag{13}$$

where $\mathcal{L}_a^{(\frac{k}{2}-1)}$ can be explicitly computed in terms of the corresponding Laguerre polynomial and e^{-2t}. Furthermore, for the zonal partition function, $Tre^{-tH_Z^{(a)}}$, we have

$$\mathcal{Z}_1^{(a)}(t) = \binom{a+(k/2)-1}{a} e^{-\frac{k\lambda t}{2}}/(1-e^{-2\lambda t})^{\frac{k}{2}}. \tag{14}$$

Also the zonal Schrödinger kernels are of the trace class which, together with their partition functions, can be described by the following explicit formulas.

$$e^{-t\mathbf{i}H_Z^{(0)}} = \big(\frac{\lambda e^{-\lambda t\mathbf{i}}}{\pi}\big)^{\frac{k}{2}} e^{\lambda_i(-\frac{1}{2}(|X|^2+|Y|^2)+e^{-2\lambda_i t\mathbf{i}}\langle X,Y+\mathbf{i}J(Y)\rangle)}, \tag{15}$$

$$e^{-t\mathbf{i}H_Z^{(a)}} = \mathcal{L}_{\mathbf{i}a}^{(\frac{k}{2}-1)}(t,X,Y))e^{-t\mathbf{i}H_Z^{(0)}}(t,X,Y), \tag{16}$$

$$\mathcal{Z}_{\mathbf{i}}^{(a)}(t) = \binom{a+(k/2)-1}{a} e^{-\frac{k\lambda t\mathbf{i}}{2}}/(1-e^{-2\lambda t\mathbf{i}})^{\frac{k}{2}} \tag{17}$$

The zonal Schrödinger-kernels are zonal fundamental solutions of the Schrödinger equation. They satisfy the Chapman-Kolmogorov identity and tend to $\delta^{(a)}$ when $t \to 0_+$.

On the zones the Wiener and Schrödinger kernels are not just of the trace class but both define complex zonal measures, namely the zonal Wiener measure $dw_{1xy}^{T(a)}(\omega)$ and the zonal Feynman measure $dw_{\mathbf{i}xy}^{T(a)}(\omega)$, on the space of continuous curves $\omega : [0,T] \to \mathbf{R}^k$ connecting two points x and y rigorously. The existence of zonal Wiener measures is not surprising. This measure exists even for the global setting. However, the trace class property is a new feature, indeed. In case of the zonal Feynman measure both the trace class property and the existence of the rigorously defined zonal Feynman measures are new features. Note that also the zonal Schrödinger kernels involve a Gauss kernel which makes these constructions well defined.

3.5 The non-periodic zones defined by Fourier-averaging.

This paragraph sketchily describes the construction of zones in the non-periodic case. (This case is not considered in the rest part of the article thus all those details are understandable without this section.) On center periodic 2-step nilpotent Lie groups the invariant subspaces W_γ, defined for a lattice point Z_γ by functions of the form $\Psi_\gamma(X,Z) = \psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma,Z\rangle}$, is identified, by the map $\Psi_\gamma \to \psi$, with function space $\mathcal{H}$ consisting of functions depending just on the X-variable. Although the zonal decomposition is established on

$\mathcal{H}$, it depends on γ and it lives, actually, on W_γ. By considering this zonal decomposition on each W_γ, it lives on $L^2(\Gamma\backslash H)$.

Such simple reduction to the X-space is not possible on non-periodic groups. Unlike in the periodic case, where the zonal functions involve just one function, $e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$, which depend on the Z-variable, the zonal functions in a zone on the non-periodic manifolds involve all the functions which depend on the Z-variable. Next we describe this construction just on the H-type groups.

In the first step, for any unit vector V_u of the Z-space, consider a complex orthonormal basis $\{Q_{V_u1}, \ldots, Q_{V_uk/2}\}$ on the complex X-space defined by the complex structure J_{V_u} which defines the complex coordinate system $\{z_{V_u1}, \ldots, z_{V_uk/2}\}$ on the X-space. This basis field must be smooth on an everywhere dense open subset of the unit Z-sphere such that it is the complement of a set of 0 measure. For given values $a_1, \ldots, a_{k/2}$ satisfying $a_1 + \cdots + a_{k/2} = a$ consider the zone, $\mathcal{H}_{V_u}^{(a_1 \ldots a_{k/2})}$, defined by $\overline{z}_{V_u1}^{(a_1)} \ldots \overline{z}_{V_uk/2}^{(a_{k/2})}$ by the Gram Schmidt orthogonalization. Then the straight zone, $\mathcal{S}^{(a_1 \ldots a_{k/2})}$, is spanned by functions of the form $\int_{\mathbf{R}^l} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)h_{V_u}^{(a_1 \ldots a_{k/2})}dV$, where $\phi(V)$ is an L^2-function defined on the Z-space $\mathbf{R}^l$ and $h_{V_u}^{(a_1 \ldots a_{k/2})}$ is eigenfunction (Itô-function in a general sense) from the corresponding zone $\mathcal{H}_{V_u}^{(a_1 \ldots a_{k/2})}$.

It can be shown that the L^2 Hilbert space on the whole group H is the direct sum of the straight zones $\mathcal{S}^{(a_1 \ldots a_{k/2})}$. The spectral investigations on these zones are much more complicated then on the zones defined for center periodic groups. For indicating the difficulties we mention that the eigenfunctions of the Klein-Gordon Laplacian Δ are of the form $\oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(V)h_{V_u}^{(a_1 \ldots a_{k/2})}dV$, where S_{R_Z} is a sphere of radius R_Z around the origin of the Z-space and $\phi(V)$ is an L^2-function defined on this sphere. This formula shows that the spectrum of the operator is continuous and each eigenvalue has infinite multiplicities. The spectral analysis with such a complicated spectrum will be developed elsewhere.

3.6 Infinities in Quantum Electrodynamics.

The problem of infinities (divergent integrals), which has been with us since the early days both of quantum field theory (cf. Heisenberg-Pauli (1929-30)) and elementary particle physics (cf. Oppenheimer (1930), Waller (1930)), is treated by *renormalization* in the current theories. This perturbative tool provides the desired finite quantities by differences of infinities. The problem of infinities is the legacy of controversial concepts such as *point mass* and

point charge of classical electron theory, which provided the first warning that a point electron will have infinite electromagnetic self-mass: the mass $e^2/6\pi ac^2$ for a surface distribution of charge with radius a blows up for $a \to 0$. Infinity appears also as infinite electromagnetic energy $\int(\mathbf{E}^2+\mathbf{B}^2)dp/8\pi$ of the Coulomb electric field $\mathbf{E}$. The infinities, related to the divergence of the summations over all possible distributions of energy/momentum of the virtual particles, mostly appear in the form of infinite traces of kernels such as the Wiener-kernel e^{-tH} or the Schrödinger kernel $e^{-tH\mathbf{i}}$.

The basic idea in the new non-perturbative approach presented in this paper is that the total quantum Hilbert space is broken up into invariant subspaces (zones) which become, so to speak, the "homes" for the zonal particles living there. The theory investigating these invariant subspaces is called de Broglie geometry. This terminology is chosen to suggest that a point, x, is a non-existing object on a zone. Rather it appears as a point spread defined by projecting the Dirac delta, δ_x, onto the zone. I. e., a point becomes a wave packet on a zone whose explicit form exhibits its very close kinship to the de Broglie waves. In a sense, de Broglie geometry ostracizes the infinities by exchanging the points for wave packets and, therefore, compels the particles to be extended.

The zones are established by means of the Landau-Zeeman operator. The biggest challenge, the Coulomb operator, has not appeared on the scene yet. By not leaving them invariant, it actually destroys the zones. This phenomenon requires a completely new attitude which is developed in the following sections.

4 Interaction with the Coulomb field

Operator defined by multiplication with the Coulomb potential function does not commute with the rest part (Landau operator) of the complete Zeeman operator but induces transmissions between the zones. The transmission from a zone into itself is called *zonal Coulomb operator*. For the explicit description of these transmission operators one should describe the eigenfunctions of the Landau operator as well as the action of the Coulomb operator on them.

4.1 Explicit eigenfunctions.

One can trace back the eigenvalue problem of $\Box_\lambda$ to the eigenvalue problem of an ordinary differential operator acting on the radial functions $f(\lambda\langle X,X\rangle)$

as follows. First let the simplest case satisfying $\lambda = 1$ be considered. By $D_\lambda \bullet f = 0$, $|Z_\lambda|^2 = k/2$, and $|J_\lambda(X)|^2 = \langle X, X\rangle$ we get

$$(\Box_\lambda F)(X) = \big(4\langle X,X\rangle f''(\langle X,X\rangle) + (2k + 4\tilde{l})f'(\langle X,X\rangle) \\ -(2m + 4((1 + \frac{1}{4}\langle X,X\rangle)f(\langle X,X\rangle))\big) H^{(\tilde{l},m)}(X). \tag{18}$$

The eigenvalue problem is reduced, therefore, to the ordinary differential operator $(L_{(\lambda=1,\tilde{l},m)}f)(t)$ defined by

$$4tf''(t) + (2k + 4\tilde{l})f'(t) - (2m + 4(\frac{k}{2} + \frac{1}{4}t))f(t). \tag{19}$$

The function $e^{-\frac{1}{2}t}$ is an eigenfunction of this operator with eigenvalue $-(4\tilde{p} + 3k)$. The general eigenfunctions are sought in the form $f(t) = u(t)e^{-\frac{1}{2}t}$, which is an eigenfunction of $L_{\tilde{l},m}$ if and only if $u(t)$ is an eigenfunction of operator $P_{(\lambda=1,\tilde{l},m)}$ defined by

$$4tu''(t) + (2k + 4\tilde{l} - 4t)u'(t) - (4\tilde{p} + 3k)u(t). \tag{20}$$

This operator has a uniquely determined polynomial eigenfunction

$$u_{(\lambda=1,n,\tilde{l},m)}(t) = t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_{n-1}t + a_n \tag{21}$$

with coefficients satisfying the recursion formulas

$$a_0 = 1 \quad , \quad a_i = -a_{i-1}(n-i)(n + \tilde{l} + \frac{1}{2}k + 1 - i)n^{-1}. \tag{22}$$

One can easily establish explicit combinatorial formula for a_i by this recursion. The eigenvalue corresponding to this polynomial is

$$\mu_{(\lambda=1,n,\tilde{l},\nu)} = -(4n + 4\tilde{p} + 3k), \text{ where } \tilde{p} = \frac{1}{2}(m + \tilde{l}). \tag{23}$$

Polynomials (21) are nothing but the Laguerre polynomials, which can be defined by the n^{th}-order polynomial eigenfunctions of the differential operator

$$\Lambda_\alpha(u)(t) = tu'' + (\alpha + 1 - t)u', \tag{24}$$

with eigenvalues $-n$. Therefore

$$P_{(\lambda=1,\tilde{l},m)} = 4\Lambda_{(\frac{1}{2}k+\tilde{l}-1)} - (4\tilde{p} + 3k). \tag{25}$$

Thus the eigenfunctions of operators (20) and (24) are the same indeed. Particularly we get that, for fixed values of $k, \tilde{l}, m$ and $\tilde{p}$, the functions $u_{(\lambda=1,n,\tilde{l},m)}$, $n = 0, 1, \ldots \infty$, form a basis in $L^2([0,\infty))$.

For a single λ, the eigenfunctions are of the form

$$u(\lambda\langle X, X\rangle)e^{-\frac{1}{2}\lambda\langle X,X\rangle}H^{(\tilde{l},m)}(\lambda^{\frac{1}{2}}X),$$

where u corresponds to $\lambda = 1$. The corresponding eigenvalue is then

$$\mu_{(\lambda,n,\tilde{l},m)} = -((4n + 4\tilde{p} + k)\lambda + 2k\lambda^2), \tag{26}$$

which statement is due to the fact that substitution $Y = \lambda^{\frac{1}{2}}X$ transforms operator $\Box_{\lambda,X}$ to $\lambda\Box_{\lambda=1,Y}$.

General eigenfunctions defined for a system $\{\lambda_1, \ldots, \lambda_r\}$ of eigenvalues are the products of eigenfunctions determined for individual λ_i's. The above explicit formulas can be established by means of these explicit eigenfunctions in the most general cases. In the 2D-case, these eigenfunctions appear in the form described in (7) and (8), where $l = \tilde{l}$ and $p = n + \tilde{p}$. The corresponding polynomials were introduced also by Itô, in the context of complex Markov processes, in the form (6). It was indicated earlier that our theory prefers the Itô polynomial technique, where the eigenfunctions are represented by products of Itô's polynomials, to the standard one in developing an effective zonal spectral investigation. For instance, in this new representation of an eigenfunction, the functions belonging to the same coordinate clearly visualize the corresponding particle in the system, while they are completely hidden in the standard representation. Next we proceed with the investigations in the 2D-case.

4.2 Transmissions and fluctuations; Zonal Coulomb fields.

By one of the definitions, the 2D zones are introduced by means of the Zeeman-Landau operator (free particle) which omits the Coulomb potential $V = Z_p e^2 r^{-1} = Qr^{-1}$ due to the nucleus. The 2D Landau operator is defined by a simple restriction of the free 3D Zeeman operator onto the (x, y)-plane, therefore, for bounded particles, also the Coulomb operator is defined by restricting the above 3D potential onto the (x, y)-plane. This is different from the 2D Coulomb potential, $Q \ln r$, which could also be considered in these investigations. The more precise arguments supporting the usage of the 3D-potential over the 2D-one are explained in the end of this section.

First note that the zones are not invariant with respect to multiplication with V. This Coulomb operator induces transmission integral operators

$V^{(a,b)} : \mathcal{H}^{(a)} \to \mathcal{H}^{(b)}$ with smooth L^2-kernels

$$V^{(a,b)}(v,w) = \int \delta_\lambda^{(b)}(v,z)V(z)\delta_\lambda^{(a)}(z,w)dz \tag{27}$$

satisfying $V^{(a,b)} = \overline{V}^{(b,a)}$. Transmission operator, $V^{(a,a)}$, mapping the zone $\mathcal{H}^{(a)}$ onto itself is called zonal Coulomb potential. Fluctuation operator on $\mathcal{H}^{(a)}$ through $\mathcal{H}^{(b)}$ is defined by $F^{(a\to b\to a)} := V^{(b,a)} \circ V^{(a,b)}$. A remarkable feature of the zonal Coulomb potential is that it turns the global interaction exhibited in the "total" Coulomb law into a local one. The same statement is valid for the zonal transmission and fluctuation operators.

In order to understand these operators more deeply, first, the matrix both of the Coulomb- and the complete Zeeman-operator in the basis formed by the eigenfunctions (6)- (8) will be explicitly described. These eigenfunctions can be parametrized by the pairs $(p, q = \upsilon)$ of zonal quantum numbers, or, by the classical quantum numbers $(n, m = p - q)$. The corresponding total, $\tau = p + q$, and azimuthal, $l = |p - q|$, quantum numbers are determined by these ones. Since V is a radial function, integral $\int H_{pq}(z)V(\sqrt{z\overline{z}})\overline{H}_{p'q'}(z)dz$ can be non-zero only for functions defined by the same magnetic quantum number $m = p - q = p' - q'$. Also note that for functions H_{nm}, when they are defined by $m \geq 0$, the n is equal to the zone index $q = \upsilon$, while, $q = \upsilon = -m + n$ holds in case of $m < 0$.

Eigenfunctions $H_{0m}, H_{1m}, H_{2m}, \ldots$ considered for a fixed magnetic number m span the so called *magnetic subspace* $\mathcal{M}_m$. Then, corresponding to $m \geq 0$ resp. $m < 0$, $\mathcal{H}^{(a)} \cap \mathcal{M}_m = \mathbf{H}_{am}$ resp. $\mathcal{H}^{(a)} \cap \mathcal{M}_m = \mathbf{H}_{(a+m)m}$ hold, where $\mathbf{H}_{nm}$ is the subspace spanned by H_{nm}. Note that $\mathbf{H}_{(a+m)m} = \mathbf{0}$ holds for $(a+m) < 0$. By the above argument, the magnetic subspaces are invariant under the action of the Coulomb operator meaning that the magnetic quantum number is invariant under the transmissions defined above. Particularly, eigenfunctions (6)-(8) are eigenfunctions both of the zonal Coulomb operators $V^{(a,a)}$ and the fluctuation operators $V^{(b,a)} \circ V^{(a,b)} = F^{(a\to b\to a)}$. Thus, both commute with the Landau operator $\Box_\lambda$.

Next the trace class properties of the zonal Coulomb and fluctuation operators will be scrutinized. The corresponding statements will be demonstrated here just on $\mathcal{H}^{(0)}$ by explicit eigenvalue computations. By formulas

$$\frac{\int_0^\infty r^{2m}e^{-\lambda r^2}dr}{\int_0^\infty r^{2m+1}e^{-\lambda r^2}dr} = \frac{2m-1}{2m}\frac{\int_0^\infty r^{2m-2}e^{-\lambda r^2}dr}{\int_0^\infty r^{2m-1}e^{-\lambda r^2}dr}, \tag{28}$$

the eigenvalues of $V^{(0,0)}$ regarding the eigenfunctions $z^m e^{-\frac{1}{2}\lambda r^2}$ are

$$Q\sqrt{\pi\lambda}\frac{1\cdot 3\ldots(2m-1)}{2\cdot 4\ldots 2m} = Q\sqrt{\pi\lambda}\frac{(2m)!}{2^{2m}(m!)^2} \approx \frac{Q\sqrt{\pi\lambda}}{\sqrt{\pi m}}, \tag{29}$$

where the estimation is computed by the Stirling formula $n! \approx \sqrt{2\pi} n^n e^{-n} \sqrt{n}$. A better approximation, $Q\sqrt{\lambda}(4m+1/3)^{\frac{1}{2}}(2m+1/3)^{-1}$, approximating the eigenvalue $Q\sqrt{\pi\lambda}$ defined for $m = 0$ by the finite value $Q\sqrt{3\lambda}$, can be established by $n! \approx \sqrt{(2n+1/3)\pi} n^n e^{-n}$. Instead of 0, the latter one approximates $0! = 1$ by $\sqrt{\pi/3} \approx 1.02333$.

Therefore, the $V^{(0,0)}$ has infinite trace and L^2-norm, however, it is in the $L^{2+\epsilon}$-class, for all $\epsilon > 0$. This statement is true for all zonal Coulomb operators and fluctuation operator $F^{(a \to b \to a)}$ satisfying $a \neq b$.

Remark. It is a natural question if, instead of the 3D-potential, the 2D-Coulomb potential should be considered in the zonal theory. The negative answer becomes clear after computing the eigenvalues (28) corresponding to the 2D Coulomb potential $Q \ln r$. Then, the recursion formula for computing the numerator is:

$$\int_0^\infty r^{2m} \ln(r) r e^{-r^2} dr = \tag{30}$$
$$\frac{1}{2}\big(r^{2m} \ln(r) e^{-r^2}_{/r=0} + 2m \int_0^\infty r^{2m-2} \ln(r) r e^{-r^2} dr + \int_0^\infty r^{2m-1} e^{-r^2} dr\big)$$
$$= \frac{1}{2}\big(\infty + 2m \int_0^\infty r^{2m-2} \ln(r) r e^{-r^2} dr + 2 \cdot 4 \ldots (2m-2)\big),$$

according to which the corresponding "eigenvalue" is $\frac{1}{2}(\infty + \cdots + \infty + \frac{1}{2m} + \frac{1}{2(2m-2)} + \ldots) = \infty + A_m$, where $A_m \to A > 0$.

The infinities and relation $A > 0$ mean that the 2D Coulomb operator does not define appropriate zonal transmission and fluctuation operators. Actually, the 2D Coulomb operator does not properly describe the physical situation considered in this paper. In fact, this operator assumes a flat 2-dimensional nucleus, whereas, in the zonal theory, the charged particle is only orbiting in a 2D plane about the nucleus which provides radiation field according to the 3D Coulomb law. This physical situation is described by means of the complex Heisenberg group representation in terms of the canonical coordinates defined on this plane. Also note that the 2D Landau operator is defined by a 3D Zeeman operator such that one drops only the 1-dimensional Laplace operator ∂_z^2 from the 3D operator. This means that the particle has zero z-kinetic energy, i. e., it is not moving into the z-direction. In other words, only the movements of the 3D particles are restricted onto planes, but they are not considered to be 2D objects. Thus really the 3D-potential is right to consider in the zonal theory. In higher dimensions the right Coulomb potential is Q^κ / r^κ.

4.3 Lamb shift

The Lamb shift, experimentally measured by Willis Lamb and Robert Retherford in 1947, is a small difference in energy between two energy levels $2s_{1/2}$ and $2p_{1/2}$ of the hydrogen atom in quantum mechanics. According to Dirac and Schrödinger theory, hydrogen states with the same n and j quantum numbers but different l quantum numbers ought to be degenerate. However, the experiment pointed out that this was not so - that the $2p_{1/2}(n = 2, l = 1, j = 1/2)$ state is slightly lower than the $2s_{1/2}(n = 2, l = 0, j = 1/2)$ state resulting in a slight shift, $\approx 1000MHz$, of the corresponding spectral line (the Lamb shift). It might seem that such a tiny effect would be deemed insignificant, but in this case that shift probed the depths of our understanding of electromagnetic theory. It became the major stimuli for renewed interest for finding effective tools which can deal with the infinities invading QED. This renewed struggle resulted the renormalization technique which is the only one, even today, in overcoming this enormous difficulty.

It was long suspected that a possible explanation might be the shift of energy levels by the interaction of the electron with the radiation field. This shift comes out infinite in all existing theories. In 1947, Hans Bethe [Be, S.J.] was the first to explain the Lamb-shift in the hydrogen spectrum. He followed the general ideas of Kramers on mass renormalization. The actual calculations are non-relativistic, whose short description is as follows.

Due to its interaction with transverse electromagnetic waves, the self-energy of an electron in a quantum state m is [H]:

$$W = -\frac{2e^2}{3\pi\hbar c^3}\int_0^\infty k \sum_n \frac{|\mathbf{v}_{mn}|^2}{E_n - E_m + k} dk = \tag{31}$$

$$-\frac{2e^2}{3\pi\hbar c^3}\int_0^\infty \Big(\sum_n |\mathbf{v}_{mn}|^2 - \sum_n \frac{|\mathbf{v}_{mn}|^2 (E_n - E_m)}{E_n - E_m + k}\Big) dk, \tag{32}$$

where $k = \hbar\omega$ is the energy of the quantum and $\mathbf{v} = \mathbf{p}/m_e = (\hbar/\mathbf{i}m_e)\nabla$ is the velocity in the non-relativistic theory. The second line decomposes into the sum, $W_0 + W'$, of two divergent integrals, where W_0 represents the change of the kinetic energy of the electron for fixed momentum, due to the fact that electromagnetic mass is added to the mass of the electron. According to the mass renormalization of Kramers, this energy should be disregarded, because this electromagnetic mass is already contained in the experimental electron mass. Therefore the relevant part of the self-energy becomes $W' = W - W_0$, which is considered to be the true shift of the levels due to

radiation interaction. Integral defining W' is still logarithmically divergent which is renormalized by cutting off the high frequencies k at $K \approx m_e c^2$. Bethe assumed that a relativistic hole-theoretic calculation would provide an explanation for this natural cutoff. After the third assumption, which considers $\ln(K/|E_n - E_m|)$ as a constant (independent of n) number, these calculations provided $W'_{2s} \approx 1040MHz$, which was in excellent agreement with the observed value of $\approx 1000MHz$.

The formulas describing the Lamb shift in a general situation are $\Delta_{Lamb} = \alpha^5 m_e c^2 \frac{k(n,0)}{4n^3}$, for $l = 0$ with $k(n,0)$ around 13 varying slightly with n, and

$$\Delta_{Lamb} = \alpha^5 m_e c^2 \frac{1}{4n^3}\big(k(n,l) \pm \frac{1}{\pi(j+\frac{1}{2})(l+\frac{1}{2})}\big), \tag{33}$$

for $l \neq 0$ and $j = l \pm \frac{1}{2}$ (inner quantum number, introduced by Sommerfeld), with $k(n,l)$ a small number (< 0.05), furthermore, $\alpha = e^2/\hbar c 4\pi\epsilon_0 \approx 7.297352568(24) \times 10^{-3} \approx 1/137$ denotes the fine structure constant.

4.4 Bethe's computation in light of the zonal theory.

On a zone the complete physics is determined by the complex Heisenberg group representation (5). On the Fock zone, where the computations are carried out, the magnetic quantum number m is equal to the azimuthal quantum number l and the normalized eigenfunctions are

$$\Psi_l = \frac{\lambda^{\frac{l+1}{2}}}{\sqrt{l!\pi}} z^l e^{-\frac{1}{2}\lambda r^2} = \psi_l e^{-\frac{1}{2}\lambda r^2}. \tag{34}$$

Thus for the velocity we have:

$$\mathbf{v}_m = (\partial_z \psi_l) e^{-\frac{1}{2}\lambda r^2} = \sqrt{l\lambda}\Psi_{l-1}, \tag{35}$$

therefore, $|\mathbf{v}_m|^2 = l\lambda = m\lambda$. Note that the only non-trivial components of $\mathbf{v}_{mn}$ resp. $E_n - E_m$ are $\mathbf{v}_{m(m-1)}$ resp. $E_{m-1} - E_m = E_m/(2m-1)$, thus Bethe's third assumption of $E_n - E_m$ being independent of n is automatically satisfied. Thus $\sum_n$ in (31) consists only of one term corresponding to $n = m-1$.

Next the implications caused by choosing the cutoff constant $K \approx m_e c^2$ on the zonal setting is explained. In the Bohr model of the hydrogen atom the Rydberg constant $Ry = hcR_\infty = \frac{m_e c^2 \alpha^2}{2}$, where $R_\infty = \frac{\alpha^2 m_e c}{4\pi\hbar}$, is defined as the energy on the innermost energy level. This level has index 1 and the electron is not resting there. The resting state corresponding to the

index 0 is not considered in that theory because it leads to false conclusions. The energy is infinity there, anyway. In our zonal model, however, the innermost energy level on $\mathcal{H}^{(a)}$ is represented by the eigenfunction $\overline{z}^a$ for which the velocity is zero: $\mathbf{v}_0^{(a)} = 0$. (This condition does not contradicts uncertainty principle because the zonal particles can not be localized at one point.) Therefore, it is reasonable to choose the Rydberg constant $R_y^{(a)}$ in the zonal theory to be the rest energy $m_e c^2$. More precisely, the spectrum for $V^{(a,a)}$ is defined such that the dimensionless spectrum, defined by assuming $Q = \lambda = 1$, is multiplied with $m_e c^2$. Since the discrete spectrum of the zonal Coulomb field is decreasing and the highest dimensionless eigenvalue is 1, the $m_e c^2$ becomes the highest energy level possible. In short, choosing cutoff constant K means choosing new Rydberg constant on the zonal setting. By the definition of $V^{(a,a)}$, the original one is $Q\sqrt{\lambda}$. It should be mention yet that the interaction theory developed here prefers $m_e c^2$ for the zonal Rydberg constant.

The greatest difficulty is created by the measure dk, measuring its domain, $[0, \infty)$, by infinity. The problems persist to exist even after introducing the cutoff constant. For instant, it still defines infinite energy summations. These difficulties seem to be originated from lacking the probabilistic feature characteristic for those measures which appear in quantum theory. We attack this problem right at this point! Bethe was compelled to use a cutoff constant along with other renormalization techniques because he had not the right finite measure allowing finite summations in store. Thus for the solution of this problem one should find the appropriate finite measure defining finite summations which directly produces the desired Lamb shift formula without any further assumption. In the zonal theory, where the multiplicity of each eigenvalue of the discrete zonal Coulomb spectrum is 1, one has a new mathematical and physical situation which makes it possible to find this measure. At this point we depart from Bethe's computations and give a new start to solving this problem.

4.5 The quantum hypothesis untying the Gordian knot.

This intricate tangle of difficulties will be untied by a quantum hypothesis similar to the Planck hypothesis. The historic Planck hypothesis concerns the amount of energy $U(\nu)d\nu$ radiated by the blackbody in the frequency range between ν and $\nu + d\nu$. By supposing equal likelihood for radiating $U(\nu)$ (equipartition principle), the old theory yielded the Rayleigh-Jeans law: $U(\nu)d\nu \sim \kappa T\nu^2 d\nu$, which contradicts the empirical curve described by the Wien law: $U(\nu)d\nu \sim \nu^3 e^{-h\nu/\kappa T} d\nu$ (κ and T denote the Boltzmann

constant and temperature respectively). The controversy arising between theory and experiment was resolved by Planck by the hypothesis that the energy attached to frequency ν is restricted to the integral multiple of the basic unit $h\nu$, i. e. $E_n = nh\nu$, where n is any positive integer number. Furthermore, the preprobability that the wall emits-absorbs an energy-quanta E_n is $\tilde{W}(n) \sim e^{-E_n/\kappa T} = e^{-nh\nu/\kappa T}$. Thus, by normalization, the probability is $W(n) = e^{-nh\nu/\kappa T}(1 - e^{-nh\nu/\kappa T})$. This hypothesis yields the Wien law.

In the present situation the quantization of the energy due to the interaction of a Landau electron with the zonal Coulomb field should be established by an adequate preprobability amplitude (finite complex measure). It is rather apparent that in integral formula (31) the electron of magnetic quantum state m interacts with the radiation field of energy level k according to the equipartition principle and not by a preprobabilistic amplitude which gives high priority for certain energies k while small chances for the other ones. By applying cutoff constants and other regularization-s, the renormalization theory works down, actually, this equipartition-amplitude to a preprobabilistic one. In our approach we write up the adequate interaction preprobability amplitude at the very beginning whose adequacy will be probed by testing if it really provides the right Lamb shift formula.

This interaction amplitude is written up for fixed dimensionless energy levels

$$\epsilon_p = (4p + 2)\lambda \quad \text{and} \quad \epsilon_B = 4\lambda^2, \quad \text{where} \quad \lambda = 1, \tag{36}$$

of the free electron resp. constant magnetic field and the dimensionless energy levels, $E_k^{(a)}$, of the zonal Coulomb field defined on a zone $\mathcal{H}^{(a)}$. In this review we give explicit computations only on the Fock zone ($a = 0$) where the dimensionless Coulomb eigenvalues are

$$E_k^{(0)} = \sqrt{\pi}\frac{(2k)!}{2^{2k}(k!)^2} = \sqrt{\pi}\frac{\Gamma(2k+1)}{2^{2k}\Gamma^2(k+1)} = \sqrt{\pi}G_k^{(0)}. \tag{37}$$

In terms of the Gamma function, such explicit formulas can be established also for the other zones. These formulas define dimensionless energies not just for the discrete values $k = 0, 1, \ldots$ but also for real numbers $k \geq 0$, allowing not just discrete but also continuous interaction amplitudes. Actually the discrete amplitude can be defined by means of the continuous one and the summations in the discrete case can be approximated by the corresponding integral defined for the continuous version. The exact form of the hypothesis prepared by these remarks is as follows.

Hypothesis for the preprobabilistic amplitude (finite complex measure) of the electromagnetic interaction. *The energy increment for an electron which is in the dimensionless energy state ϵ_p resp. ϵ_B and interacts with a dimensionless zonal radiation field $V^{(a,a)}$ can be determined by the preprobabilistic amplitude (finite complex measure)*

$$\mathcal{A}^{(a)}_{\mathbf{i}}(\epsilon_.,k)dk = \sqrt{m_ec^2}\alpha^{\frac{5}{2}}\frac{1}{2}de^{-\frac{1}{2}\sqrt{\pi}\epsilon_.E^{(a)}_k\mathbf{i}} \tag{38}$$

$$= \sqrt{m_ec^2}\alpha^{\frac{5}{2}}\frac{1}{2}(-\frac{\pi}{2}\epsilon_.G^{(a)\prime}_k\mathbf{i})e^{-\frac{1}{2}\pi\epsilon_.G^{(a)}_k\mathbf{i}}dk$$

$$= \sqrt{m_ec^2}\mathcal{B}^{(a)}_{\mathbf{i}}(\epsilon_.,k)dk, \tag{39}$$

where $\epsilon_.$ stands for ϵ_p or ϵ_B, such that the amplitude for the energy shift, due to the interaction with the zonal Coulomb field, is the finite value $\Sigma^{(a)}_{\mathbf{i}}(\epsilon_.) = \int_{k=0}^{\infty}E^{(a)}_k\mathcal{A}^{(a)}_{\mathbf{i}}(\epsilon_.,k)dk$ and, therefore, the corresponding Lamb shift (energy increment) is $\Delta^{(a)}_{Lamb} = \Sigma^{(a)}_{\mathbf{i}}(\epsilon_.)\overline{\Sigma}^{(a)}_{\mathbf{i}}(\epsilon_.)$.

The discrete density $\mathcal{D}_{\mathbf{i}}(\epsilon_.,k)$ is defined by restricting the continuous density $\mathcal{A}^{(a)}_{\mathbf{i}}(\epsilon_.,k)$ to the discrete set $k = 0,1,2,\ldots$ of numbers. Then the summations corresponding to the above integrals can be approximated by the above integrals.

The scalar density, $\mathcal{A}^{(a)}_1(\epsilon_.,k)dk$, is defined by omitting $\mathbf{i}$ from the above formulas. This density can be used for further studying Bethe's computations (these details are omitted).

In the density formula (38) all quantities are dimensionless except m_ec^2 which is the energy of the resting electron. The fine-structure constant $\alpha \approx 1/137$ is the fundamental physical constant characterizing the strength of the electromagnetic interaction. It is a dimensionless quantity, and thus its numerical value is independent of the system of units used. It can be thought of as the square of the ratio of the elementary charge to the Planck charge. For any arbitrary length s used in formulas $2\pi s = \lambda = \frac{c}{\nu}$ of classical quantum theory, the α is the ratio of two energies. Its value cannot be predicted by the theory, and has to be inserted based on experimental results. In fact, it is one of the twenty-odd "external parameters" in the Standard Model of particle physics. This also means that the above Hypothesis is independent from the other ones of quantum theory which has to be objected to experimental testing. Below we show that this Hypothesis implies the Lamb shift formula, thus this testing is already done by the Lamb-Retherford experiment. Thus the positively tested Hypothesis lifts out α from the set of the twenty-odd external parameters and turns it into an internal, equal partner to the Planck constant.

At this point of argumentation it is appropriate to quote from the preface written to the "Pauli Lectures on Physics, Volume 1." by the last assistant of Pauli: "For Pauli the central problem of electrodynamics was the field concept and the existence of an elementary charge which is expressible by the fine structure constant $e^2/mc = 1/137$. This fundamental pure number had greatly fascinated Pauli, as can be seen from the list of references to his work assembled in the appendix. For Pauli the explanation of the number 137 was the test of a successful field theory, a test which no theory has passed up to now. This number 137 transcendented into a magic symbol at Pauli's death. When I visited Pauli in the hospital, he asked me with concern whether I had noticed his room number: 137! It is in this room that he died a few days later. Charles P. Enz, Geneva. 17 November 1971."

This quote inspires the following reformulation of the above ideas developed for the complex plane $\mathbf{C}$.

Reformulated Hypothesis in terms of Dimensionless Quantum Theory ($D_{less}QTh$). *The dimensionless quantum theory is based on the complex Heisenberg algebra representation (5) defined for $\lambda = 1$. The irreducible subspaces, $\mathcal{H}^{(a)}$, of this reducible representation are the so called dimensionless Zeeman zones among which $\mathcal{H}^{(0)}$ is the Fock space. The dimensionless Hamiltonian associated with this representation is*

$$-\Box = -(\Delta_X - 2\mathbf{i}D\bullet - 4(1+\frac{1}{4}|X|^2)), \tag{40}$$

where $D\bullet$ is defined by the complex structure J involved to mathematical modeling. This operator is the sum of the dimensionless Landau operator and a constant Hamiltonian belonging to the constant magnetic field. The dimensionless spectrum of $\Box$ is $-((4p+2)+4)$, where $p = n+l$ is established earlier as a quantum number for the Landau electron and 4 belongs to the constant magnetic field. Each eigenvalue in this spectrum has infinite multiplicity and the the zones are established by sorting out the eigenfunctions such that, on a zone, each multiplicity becomes 1 and any two zones are isospectral.

The dimensionless Coulomb potential is $1/r$ which defines a dimensionless Coulomb integral operator $V^{(a,a)}$ for each zone. This zonal Coulomb operator commutes with the Landau operator and operator $-\Box^{(a)} - V^{(a,a)}$ is the complete dimensionless zonal Zeeman Hamiltonian. The dimensionless Coulomb eigenvalues on the Fock zone are described by (37). In terms of Gamma function and index (a), they can be explicitly computed for all zones.

The dimensionless elementary charge is defined by $\aleph = \left(\frac{1}{4}\alpha^5\right)^{\frac{1}{6}} \approx \frac{1}{76}$ (see the explanation below) and the dimensionless preprobabilistic amplitude

$\mathcal{B}^{(a)}_{\mathbf{i}}(\epsilon_.,k)dk$ *is defined in terms of the cubed elementary charge* $\frac{1}{2}\alpha^{\frac{5}{2}} = \aleph^3$ *and an elementary density in formula (39). Then the dimensionless energy due to the interaction between the electron, having the quantum numbers* $\epsilon_p = 4p + 2$ *resp.* $\epsilon_B = 4$*, and the zonal Coulomb field can be determined as follows. The dimensionless amplitude for this energy is* $\beta^{(a)}_{\mathbf{i}}(\epsilon_.) = \int_{k=0}^{\infty} E^{(a)}_k \mathcal{B}^{(a)}_{\mathbf{i}}(\epsilon_.,k)dk$ *and the total amplitude,* $\beta^{(a)}_{\mathbf{i},total}$*, is the sum of the two amplitudes. The corresponding dimensionless Lamb shifts (dimensionless energy increments due to interactions) are* $\Delta^{(a)}_{D_{less}Lamb}(\epsilon_.) = \beta^{(a)}_{\mathbf{i}}(\epsilon_.)\overline{\beta}^{(a)}_{\mathbf{i}}(\epsilon_.)$ *and* $\Delta^{(a)}_{D_{less}Lamb}(total) = \beta^{(a)}_{\mathbf{i},total}\overline{\beta}^{(a)}_{\mathbf{i},total}$*, where the last formula is based on the independence of the Hamiltonian of the constant magnetic field from the rest part of the Hamiltonian. (This independence is exhibited by the commutativity of these operators.) For an electron in quantum state* $\epsilon_. = \epsilon_p$*, or,* ϵ_B*, the true Lamb shift having complete physical dimensions is*

$$\Delta^{(a)}_{Lamb}(\epsilon_.) = m_e c^2 \Delta^{(a)}_{D_{less}Lamb}(\epsilon_.), \tag{41}$$

$$\Delta^{(a)}_{Lamb}(total) = m_e c^2 \Delta^{(a)}_{D_{less}Lamb}(total). \tag{42}$$

Instead of one electron one can consider also a zonal charge-field, defined by the density $\rho^{(a)}_e(z)$*, such that at each point* $z \in \mathbf{C}$ *the charge is in the total quantum state determined by* ϵ_p *resp.* ϵ_B*. Then the energy due to the interaction of the charge-field with the zonal Coulomb field is*

$$\Delta^{(a)}_{Lamb}(total) = \big(\int \rho^{(a)}_e c^2 dz\big)\Delta^{(a)}_{D_{less}Lamb}(total). \tag{43}$$

In Bethe's formula, obtained for the Lamb shift, constant $\frac{1}{4}\alpha^5 m_e c^2$ appears in the form $\frac{1}{2}\alpha^3 Ry$, where $Ry = hc/\lambda = hc/(c\nu) = h\nu$ is the energy on the innermost energy level of the hydrogen atom. This constant includes α by the following computations:

$$R_\infty = \frac{\alpha^2 m_e c}{4\pi\hbar}, \quad Ry = hcR_\infty = \frac{m_e c^2 \alpha^2}{2}.$$

The Rydberg constant is defined by means of the Bohr model of hydrogen atom where the innermost level has index 1 and the electron is not resting there. Considering the resting state with index 0 leads to false conclusions in that model. In our zonal model, however, the innermost energy level on $\mathcal{H}^{(a)}$ is represented by the eigenfunction $\overline{z}^a$ for which the velocity is zero: $\mathbf{v}^{(a)}_0 = 0$. Therefore, in the zonal theory, the Rydberg constant $R^{(a)}_y$ can be defined by the rest energy $m_e c^2$. If one considers Bethe's formula as a natural

pattern and assumes that the Lamb shift constant should appear in terms of an elementary charge $\aleph$ in the form $\aleph^6 R_y^{(a)}$ then this elementary charge must satisfy $\aleph^6 = \frac{1}{4}\alpha^5$. In this interpretation both the electron and the nucleus are supposed to be charged with the elementary charge. This is the major motivation for defining the elementary charge in the form described in the Hypothesis.

4.6 Testing the Hypothesis by the Stirling approximations.

The real test of the Hypothesis is if it really provides the Lamb shift formula (33). The computations are demonstrated below on the Fock zone and, instead of (38), an approximating density defined by the Stirling approximation of $n!$ will be used for computations. To avoid divergence at 0, the second Stirling formula is the appropriate one for defining $E_k^{(0)} \approx (4k + 1/\pi)^{\frac{1}{2}}/(2k + 1/\pi) = S_k$. As opposed to the original version, where 3 is used instead of π, this Stirling formula approximates 0! by 1.

Thus the approximating Stirling density for $\epsilon_p = 4p + 2 = 4l + 2 = \epsilon_l$ is

$$de^{-\sqrt{\pi}(2l+1)S_k\mathbf{i}} = -\sqrt{\pi}(2l+1)S_k'\mathbf{i}e^{-\sqrt{\pi}(2l+1)S_k\mathbf{i}}dk \tag{44}$$

and the amplitude for the shift of the self energy of the electron due to the interaction with the zonal Coulomb field can be found by the following computations:

$$\sigma_{\mathbf{i}l}^{(0)} = \int_0^\infty S_k de^{-\sqrt{\pi}(2l+1)S_k\mathbf{i}} = \int_0^\infty S_k\big(e^{-\sqrt{\pi}(2l+1)S_k\mathbf{i}}\big)'dk = \tag{45}$$

$$= -\sqrt{\pi}e^{-(2l+1)\pi\mathbf{i}} - \int_{k=0}^\infty S_k' e^{-\sqrt{\pi}(2l+1)S_k\mathbf{i}}dk =$$

$$= \sqrt{\pi} + \frac{1}{\sqrt{\pi}(2l+1)\mathbf{i}}\int_0^\infty \big(e^{-\sqrt{\pi}(2l+1)S_k\mathbf{i}}\big)'dk =$$

$$= \sqrt{\pi} + \frac{1}{\sqrt{\pi}(2l+1)\mathbf{i}}(1 - e^{-(2l+1)\pi\mathbf{i}}) = \sqrt{\pi} - \frac{\mathbf{i}}{\sqrt{\pi}(l+\frac{1}{2})},$$

$$\Delta_{Lamb}^{(0)}(\epsilon_p) = m_e c^2\alpha^5\frac{1}{4}\sigma_{\mathbf{i}l}^{(0)}\overline{\sigma}_{\mathbf{i}l}^{(0)} = m_e c^2\alpha^5\frac{1}{4}(\pi + \frac{1}{\pi(l+\frac{1}{2})^2}) \tag{46}$$

The Hamilton operator of the constant magnetic field commutes with the complete operator, thus the complete amplitude is the sum of $\sigma^{(0)}_{\mathbf{i}l}$ and $\sigma^{(0)}_{\mathbf{i}B}$. The above computation provides the latter amplitude by substituting $(2l+1)$ by 2. Then we have

$$\sigma^{(0)}_{\mathbf{i}B} = -\sqrt{\pi}, \quad \sigma^{(0)}_{total} = \sigma^{(0)}_{\mathbf{i}l} + \sigma^{(0)}_{\mathbf{i}B} = -\frac{\mathbf{i}}{\sqrt{\pi}(l+\frac{1}{2})}, \tag{47}$$

$$\Delta^{(0)}_{total} = \frac{m_e c^2 \alpha^5}{4\pi(l+\frac{1}{2})^2}. \tag{48}$$

Although term $\pi^{-1}(l+\frac{1}{2})^{-2}$ can be written in form

$$\frac{1}{\pi(l+\frac{1}{2})^2} = \frac{1}{2\pi(l+1)(l+\frac{1}{2})^2} + \frac{1}{\pi(l+1)(l+\frac{1}{2})},$$

where the first term on the right side is a decreasing sequence determining small numbers < 0.03536 for $l \geq 1$, this first term can not be identified with $k(n,l)$ of the experimental formula (33). The $k(n,l)$ is due to other interactions not considered in this paper. The formula established here concerns just the interaction with the Coulomb field, which is the main contribution to the Lamb shift. Even this main term is established in (33) by considering the inner quantum number j. This quantum number has net appeared on the scene yet and will furnished into the zonal theory after introducing the spin concept. This is why $(l+\frac{1}{2})^2$ appears in the place of $\frac{1}{\pi(l+\frac{1}{2})(j+\frac{1}{2})}$ in the scalar theory presented in this paper. The discrepancies regarding the radial quantum number n are due to the discrepancies between the classical and zonal radial quantum numbers. The zonal one is included into $p = n + l$ and it is zero on the Fock zone. Thus $1/n^3$ in formula (33) does not make any sense there. Furthermore, the dependence on n seems to be just quadratic on the zonal setting. However, the explicit formulas of Lamb shift on the higher order zones reveal a higher order dependence on the radial quantum number.

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